

Stochastic resonance in the driven Ising model on small-world networks

H. Hong*

Korea Institute for Advanced Study, Seoul 130-012, Korea

Beom Jun Kim

Department of Molecular Science and Technology, Ajou University, Suwon 442-749, Korea

M.Y. Choi

Department of Physics, Seoul National University, Seoul 151-747, Korea

We investigate the stochastic resonance phenomena in the field-driven Ising model on small-world networks. The response of the magnetization to an oscillating magnetic field is examined by means of Monte Carlo dynamic simulations, with the rewiring probability varied. At any finite value of the rewiring probability, the system is found to undergo a dynamic phase transition at a finite temperature, giving rise to double resonance peaks. While the peak in the ferromagnetic phase grows with the rewiring probability, that in the paramagnetic phase tends to reduce, indicating opposite effects of the long-range interactions on the resonance in the two phases.

PACS numbers: 89.75.Hc, 05.40.-a, 75.10.Hk

I. INTRODUCTION

It has been known that a periodically modulated bistable system may display stochastic resonance (SR), arising from the cooperative interaction between random noise and periodic modulation [1]. Namely, when a system with an energetic activation barrier is subject to periodic but weak driving, the inherent thermal stochastic noise can enhance the signal out of the system rather than weaken it. Such SR phenomena, which have various practical applications [2], have been extensively investigated in a variety of systems: In parallel to studies of systems with relatively small numbers of degrees of freedom, the research focus has been shifting to extended complex systems with many degrees of freedom, where interesting collective dynamic behavior can emerge [3, 4]. In those studies of SR in extended systems, the underlying connection topology of dynamic variables has usually been assumed to be regular. However, recent studies of computer networks, neuronal networks, biochemical networks, and even social networks, have revealed that many real systems in nature possess quite complex structures, which can be described neither by regular networks nor by completely random networks [5].

In this paper we consider an Ising model on Watts and Strogatz (WS) type small-world networks [6] and study the SR behavior in the presence of temporally oscillating external magnetic fields. The WS network is characterized by a short characteristic path length and high clustering [6], both of which are commonly observed properties of real networks in nature. Accordingly, we believe that the study of SR in an extended system can be made more realistic if one uses the WS network as the under-

lying topology.

There are five sections in this paper: Section II introduces the system, i.e., the Ising model, driven by oscillating magnetic fields, on small-world networks. Defined in Sec. III is the dynamic order parameter, which describes conveniently the dynamic phase transition. The relaxation behavior is examined and the relaxation time is computed. Section IV is devoted to the stochastic resonance phenomena, characterized by double resonance peaks in the occupation ratio. The effects of long-range interactions on the resonance are investigated, revealing opposite trends in the ferromagnetic and paramagnetic phases. Finally, the main result is summarized and discussed in Sec. V.

II. ISING MODEL ON SMALL-WORLD NETWORKS

The WS network in the present paper is constructed following Ref. [6]: First, a one-dimensional regular network with only local connections (of range k) is constructed with the periodic boundary condition. Next, each local link is visited once, and with the rewiring probability P removed and reconnected to a randomly chosen node. After the whole sweep of the entire network, the average number of shortcuts in the network of size N is given by NPk . Throughout this paper, the interaction range k is set equal to two for convenience; longer ranges ($k > 2$) are not expected to lead to any qualitative difference. After the WS network is built as above, an Ising spin is put on every node, and an edge (or a link) connecting two nodes is regarded as the coupling between the two spins at the two nodes.

The Hamiltonian for the field-driven Ising model on

*Electronic address: hhong@kias.re.kr

the WS network thus reads

$$H = -\frac{J}{2} \sum_i \sum_{j \in \Lambda_i} \sigma_i \sigma_j - h(t) \sum_i \sigma_i, \quad (1)$$

where J is the coupling strength, $\sigma_i (= \pm 1)$ is the Ising spin at node i , the neighborhood Λ_i of i stands for the set of nodes connected to i (via either local edges or shortcuts), and an oscillating magnetic field $h(t) = h_0 \cos \Omega t$ is applied with the driving amplitude h_0 and frequency Ω . We perform Monte Carlo (MC) dynamic simulations, employing the heat bath algorithm [7] and measuring the time t in units of the MC time step. For thermalization, we start from sufficiently high temperatures and lower the temperature T slowly with the increment $\Delta T = 0.02$ (in units of J/k_B with the Boltzmann constant k_B). The driving amplitude and frequency are mostly taken to be $h_0 = 0.1$ and $\Omega = 0.1$ although different frequencies are also considered. While simulations are performed at a given temperature, the data from the first 4×10^4 MC steps are discarded, which turns out to be sufficient for stationarity, and measurements are made for next 10^4 MC steps. Networks of various sizes, up to $N = 6400$, are constructed as described above, and averages are performed over 100 different network realizations.

In the absence of the long-range interaction ($P = 0$), the network structure reduces to that of the one-dimensional regular network with only local couplings. Accordingly, when $P = 0$, the driven Ising model described by Eq. (1) as well as the undriven model ($h_0 = 0$) should not exhibit long-range order at finite temperatures. For $P \neq 0$, on the other hand, it has been found that the (undriven) Ising model displays ferromagnetic order at finite temperatures [8]. This suggests that the driven Ising model in Eq. (1) should undergo a dynamic phase transition at a finite temperature unless $h(t)$ is too large. When all nodes are fully connected, Eq. (1) describes the infinite-range Ising model, where double SR peaks have been observed and argued to be a generic property of the system with a continuous dynamic phase transition [9]. In view of this, similar double SR peaks are naturally expected when $P \neq 0$; this contrasts with the case $P = 0$, where the absence of a dynamic phase transition implies the emergence of just a single SR peak [10].

III. DYNAMIC PHASE TRANSITION

The dynamic phase transition in the system can be conveniently described by the dynamic order parameter. We first measure the magnetization

$$m(t) \equiv \frac{1}{N} \sum_{i=1}^N \sigma_i, \quad (2)$$

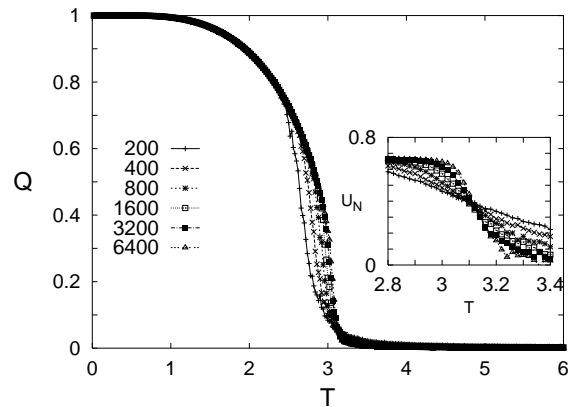


FIG. 1: Dynamic order parameter Q versus the temperature T in the WS network of size $N = 200, 400, 800, 1600, 3200$, and 6400 , at the rewiring probability $P = 0.5$. Inset: Dynamic fourth-order cumulant U_N has a unique crossing point at $T_c \approx 3.13$ (in units of J/k_B).

and take the time average of $m(t)$ during the n -th period of $h(t)$, to obtain

$$Q_n \equiv \frac{\Omega}{2\pi} \int_{t_n}^{t_{n+1}} dt m(t) \quad (3)$$

with $t_n \equiv 2\pi n/\Omega$. The dynamic order parameter is then defined to be

$$Q \equiv \lim_{n \rightarrow \infty} Q_n, \quad (4)$$

which takes different values below and above the critical temperature T_c : Figure 1 shows that $Q \neq 0$ in the ferromagnetic phase at $T < T_c$ and $Q = 0$ in the paramagnetic phase at $T > T_c$, with $T_c \approx 3$ for $P = 0.5$. In order to determine T_c more precisely, we measure the dynamic fourth-order cumulant [11, 12]

$$U_N \equiv 1 - \frac{[\langle Q_n^4 \rangle]}{3[\langle Q_n^2 \rangle]^2}, \quad (5)$$

where $\langle \dots \rangle$ and $[\dots]$ denote the time average and the average over different network realizations, respectively. At $T = T_c$, the cumulant U_N should have a unique value regardless of the size of the system; this yields the estimation $T_c \approx 3.13$, as shown in the inset of Fig. 1.

To investigate the time relaxation behavior, we begin with the initial condition $m(t=0) = 1$ (i.e., $\sigma_i = 1$ for all i), and measure Q_n in Eq. (3) as a function of t_n during MC simulations. As time proceeds, Q_n approaches the dynamic order parameter [see Eq. (4)] with the value $Q = 0$ and $Q \neq 0$ in the high- and low-temperature phases, respectively. We find that the relaxation of Q_n is very well described by the exponential form, $Q_n - Q \propto e^{-t_n/\tau}$ with $t_n = 2\pi n/\Omega$, which defines the relaxation time τ [13]. Figure 2 shows the relaxation behavior for $P = 0.5$ and 0.8 at $T = 3.5$; it is observed

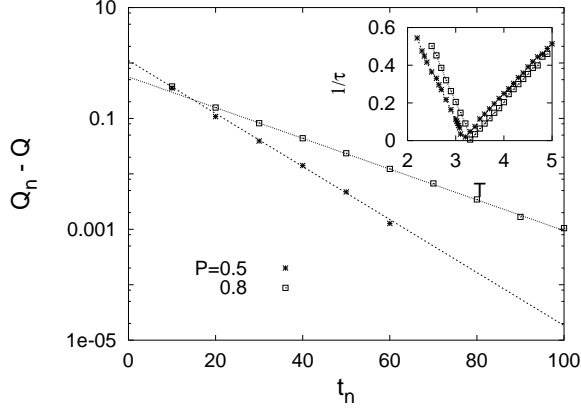


FIG. 2: Relaxation of the dynamic order parameter. $Q_n - Q$ versus t_n at temperature $T = 3.5$ is plotted in the semilog scale for the rewiring probability $P = 0.5$ and 0.8 . As the measurement time t_n is increased, Q_n is shown to approach its steady state value Q . Dashed lines represent the least-square fit to the exponential-decay form, $Q_n - Q \propto e^{-t_n/\tau}$, from which one can obtain the relaxation time τ . Inset: Inverse of the relaxation time $1/\tau$ versus the temperature T .

that the exponential decay form describes the simulation results very well. We repeat the same procedure at other temperatures and obtain the temperature dependence of the relaxation time, shown in the inset of Fig. 2. As expected from the existence of the finite-temperature dynamic phase transition, the relaxation time diverges near the dynamic transition point ($T_c \approx 3.13$ and 3.20 for $P = 0.5$ and 0.8 , respectively), and decreases as we move away from T_c in both directions [9].

IV. STOCHASTIC RESONANCE

In this section we study how the long-range interactions influence SR by varying the rewiring probability P . The SR behavior is conveniently captured by the occupancy ratio R , first introduced in Ref. [4] and defined to be the average fraction of the spins in the direction of the external field [14]:

$$R \equiv \left\langle \frac{\text{number of spins in the direction of } h(t)}{\text{total number of spins}} \right\rangle. \quad (6)$$

In other words, R measures how many spins follow the oscillating magnetic field. It is easy to understand that R has the value $1/2$ in both low- and high-temperature limits (see, e.g., Ref. [9]) and becomes larger near the SR temperature, reflecting that more spins follow the external driving.

In Fig. 3, R in the system with $P = 0.5$ is plotted as a function of the temperature for various sizes N . The double SR peaks are clearly exhibited, with the dip emerging in the vicinity of the dynamic phase transition temperature T_c determined from the crossing of U_n . As the

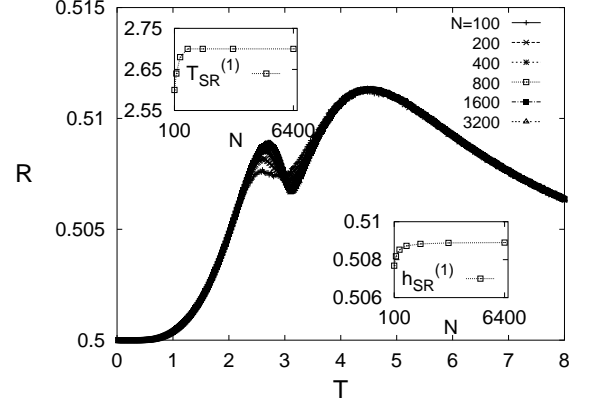


FIG. 3: Size effects of the occupancy ratio R versus the temperature T for $P = 0.5$. The finite-size effects are stronger for the SR peak at the lower resonance temperature but become negligible for $N \gtrsim 800$. Insets: Size-dependence of the SR temperature $T_{SR}^{(1)}$ and of the height $h_{SR}^{(1)}$ of the first resonance peak.

size N is increased, the first SR peak at the lower resonance temperature $T_{SR}^{(1)}$ becomes sharper and apparently saturates for $N \gtrsim 800$, displaying negligible finite-size effects. The SR temperature $T_{SR}^{(1)}$ and the height $h_{SR}^{(1)}$ of the first resonance peak are displayed in the insets of Fig. 3, manifesting that there are no substantial size effects for $N \gtrsim 800$. For the second SR peak at the higher resonance temperature $T_{SR}^{(2)}$, finite-size effects are insignificant in both the height $h_{SR}^{(2)}$ and the SR temperature $T_{SR}^{(2)}$. We thus conclude that the double SR peaks are not merely finite-size effects, and obtain $T_{SR}^{(1)} \approx 2.70$ and $T_{SR}^{(2)} \approx 4.46$ from the positions of the two SR peaks in Fig. 3.

In Ref. [9], the positions of SR peaks in the infinite-range Ising model have been analytically obtained from the time-scaling matching condition: The intrinsic time scale given by the relaxation time τ should match the extrinsic time scale of the external driving. Furthermore, since τ diverges only at T_c and decreases as the temperature T is raised or lowered from T_c , the presence of double SR peaks should be a general property of the system with a finite-temperature continuous phase transition. However, the specific form of the matching condition for the infinite-range Ising model, $\tau = 1 + \sqrt{1 + \Omega^{-2}}$, found in Ref. [9] may not hold for the Ising model on the WS network studied in this work. Here we reveal the relation between the positions of double SR peaks and the relaxation time in the following manner: We first draw a vertical line at $T = T_{SR}^{(2)}$ (≈ 4.46) in the inset of Fig. 2 and locate the crossing point with the plot for $1/\tau$, from which a horizontal line is drawn. This horizontal line corresponds to the time-scale matching condition and its crossing point with the lower temperature branch of the $1/\tau$ plot then yields the position of the first SR peak $T_{SR}^{(1)}$.

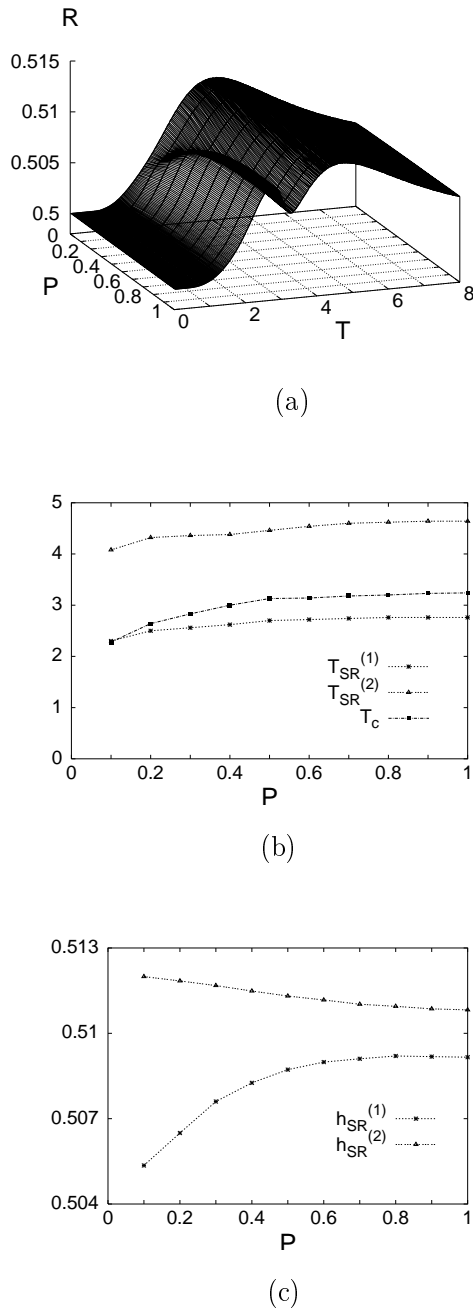


FIG. 4: (a) Occupancy ratio R versus the temperature T and the rewiring probability P in the system of size $N = 800$. (b) Lower and upper resonance temperatures $T_{SR}^{(1)}$ and $T_{SR}^{(2)}$ as well as the dynamic transition temperature T_c versus P . (c) Heights $h_{SR}^{(1)}$ and $h_{SR}^{(2)}$ of the resonance peaks versus P .

This gives $T_{SR}^{(1)} \approx 2.70$, which is in excellent agreement with the value obtained from the first peak of R in Fig. 3.

The behavior of R with the temperature T at various values of the rewiring probability P is shown in Fig. 4. When long-range interactions are not present ($P = 0$), the Ising model does not establish ferromagnetic order at finite temperatures, thus not displaying a finite-temperature phase transition. Then expected is a single

SR peak instead of two peaks [9], which is confirmed in Fig. 4. As P is increased, the system behaves more similarly to the mean-field system, and exhibits clear double SR peaks, which, in the mean-field (infinite-range) limit, have equal heights [9]. Note that the maximally connected case ($P = 1$) of the WS network has $O(N)$ long-range connections while the infinite-range system has $O(N^2)$ connections. This explains why the double peaks of R at $P = 1$ do not have equal heights in Fig. 4.

The SR peaks in Fig. 4 may be located from the condition $dR/dT = 0$ except for the first resonance peak at $P = 0.1$, where we have used the condition that dR/dT has a minimum positive value. In the insets of Fig. 4, the obtained SR temperatures and peak heights are shown as functions of P . It is noteworthy that as P is increased the height $h_{SR}^{(1)}$ of the first peak also grows, whereas $h_{SR}^{(2)}$ reduces. In other words, addition of long-range interactions tends to enhance SR in the ferromagnetic phase, while the same addition suppresses SR in the paramagnetic phase. To understand these conflicting effects, which appear somewhat counterintuitive, we notice that as P is increased, the critical temperature T_c shifts toward higher temperatures, reducing the relative (temperature) distance to $T_{SR}^{(2)}$ in comparison with that to $T_{SR}^{(1)}$. In the paramagnetic phase, the correlation length should increase with P since $T_{SR}^{(2)}$ becomes closer to T_c ; this leads more spins to be correlated and thus resistive to the change of the external driving field, resulting in the suppression of SR. In the ferromagnetic phase, on the other hand, a larger number of shortcuts (i.e., long-range connections) increases the distance to T_c , yielding a shorter correlation length. Accordingly, the spins are allowed to follow better the external driving, and the SR is enhanced. It is also to be noted that SR in the present system tends to saturate for $P \gtrsim 0.7$ and that almost the same SR as the random network ($P = 1.0$) can be achieved with relatively small numbers of shortcuts. Similar enhancing and saturating tendency has also been found in the synchronization behavior of the coupled oscillators on small-world networks [15] as well as in the standard (single-peak) SR behavior of the system of coupled bistable elements [16].

We have also considered different driving frequencies and investigated how the occupancy ratio behaves. Figure 5 displays the behavior of R depending on the temperature T for various driving frequencies. The interval between the two resonance peaks is observed to grow as the driving frequency is increased, which is consistent with the obtained relation between the relaxation time and the driving frequency. On the other hand, the heights of the two peaks tend to reduce with the driving frequency; eventually, the lower peak in the ferromagnetic phase disappears, which reflects that at high driving frequencies there exists only one crossing point between the relaxation time and the driving frequency.

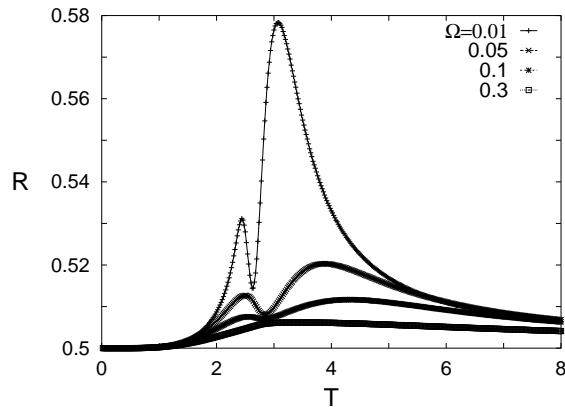


FIG. 5: Occupancy ratio R versus the temperature T for the rewiring probability $P = 0.3$ and several driving frequencies. While the interval between the two peaks tends to grow with the driving frequency Ω , the heights of the peaks in general reduce with Ω . At high frequencies ($\Omega = 0.3$), the lower peak in the ferromagnetic phase is observed to disappear.

V. SUMMARY

We have examined the stochastic resonance phenomena in the Ising model, driven by oscillating magnetic fields, on small-world networks. Double resonance peaks have been found to develop as the rewiring probability is increased, and explained in terms of the diverging time scale at the dynamic phase transition. It has been demonstrated that the resonance behavior essentially the same as that of the random network can be achieved with relatively small numbers of shortcuts. The most

interesting finding in our system is that while stochastic resonance is enhanced with the number of long-range connections in the ferromagnetic phase, it is suppressed in the paramagnetic phase. As an interpretation based on analogy, we suggest to consider the opinion formation in a social system: an instructor and a group of students. More long-range connections in this analogy correspond to stronger interactions, i.e., more active discussions, between students far away, and thus help the group behave as a whole with a majority of students having the same opinion (this situation is closely related to the enhanced synchronization for larger P in Ref. [15]). However, this opinion developed from active discussions among students may not necessarily be the opinion preferred by the instructor of the class. The enhanced SR peak in the ferromagnetic phase corresponds to the situation that more interactions among students help the class follow the instructor, with the students having already an identical opinion (whatever it is). On the other hand, the suppression of SR in the paramagnetic phase is interpreted as the situation that more interactions make the students, among whom an agreement is not reached, not so obedient to the instructor.

Acknowledgments

H.H. thanks J. Lee for providing the privilege of using the computing facility Gene. B.J.K. acknowledges the support of Ajou Univ. for the year 2002. M.Y.C. thanks the Korea Institute for Advanced Study for hospitality during his visit, where this work was performed, and acknowledges the partial support from the Korea Research Foundation through Grant No. 2000-015-DP0138.

-
- [1] B. McNamara and K. Wiesenfeld, *Phys. Rev. A* **39** 4854 (1989); P. Jung and P. Hänggi, *Phys. Rev. A* **44**, 8032 (1991); L. Gammaitoni, P. Hänggi, P. Jung, and F. Marchesoni, *Rev. Mod. Phys.* **70**, 223 (1998).
 - [2] S. Fauve and F. Heslot, *Phys. Lett. A* **97**, 5 (1983); M.H. Devoret, J.M. Martinis, D. Esteve, and J. Clarke, *Phys. Rev. Lett.* **53**, 1260 (1984); B. McNamara, K. Wiesenfeld and R. Roy, *ibid.* **60**, 2626 (1988).
 - [3] A.R. Bulsara and G. Schmerla, *Phys. Rev. E* **47**, 3734 (1993); F. Marchesoni, L. Gammaitoni, and A.R. Bulsara, *Phys. Rev. Lett.* **76**, 2609 (1996); P.C. Gailey, A. Neiman, J.J. Collins, and F. Moss, *ibid.* **79**, 4701 (1997); S. Kim, S.H. Park, and H.-B. Pyo, *ibid.* **82**, 1620 (1999); H. Hong and M.Y. Choi, *Phys. Rev. E* **62**, 6462 (2000).
 - [4] J.F. Lindner, B.K. Meadows, W.L. Ditto, M.E. Inchiosa, and A.R. Bulsara, *Phys. Rev. Lett.* **75**, 3 (1995).
 - [5] For reviews of complex networks, see, e.g., *Science* **284**, 79-109 (1999); M.E.J. Newman, *J. Stat. Phys.* **101**, 819 (2000); S.H. Strogatz, *Nature* **410**, 268 (2001); R. Albert and A.-L. Barabási, *Rev. Mod. Phys.* **74**, 47 (2002); S.N. Dorogovtsev and J.F.F. Mendes, *Adv. Phys.* (in press).
 - [6] D.J. Watts and S.H. Strogatz, *Nature (London)* **393**, 440 (1998).
 - [7] See, e.g., M.E.J. Newman and G.T. Barkema, *Monte Carlo Methods in Statistical Physics* (Oxford, New York, 1999).
 - [8] M. Gitterman, *J. Phys. A: Math. Gen.* **33**, 8373 (2000); A. Barrat and M. Weigt, *Eur. Phys. J. B* **13**, 547 (2000); A. Pękalski, *ibid.* **64**, 057104 (2001); H. Hong, B.J. Kim, and M.Y. Choi (unpublished).
 - [9] B.J. Kim, P. Minnhagen, H.J. Kim, M.Y. Choi, and G.S. Jeon, *Europhys. Lett.* **56**, 333 (2001).
 - [10] J.J. Brey and A. Prados, *Phys. Lett. A* **216**, 240 (1996).
 - [11] S.W. Sides, P.A. Rikvold, and M.A. Novotny, *Phys. Rev. Lett.* **81**, 834 (1998).
 - [12] K. Binder and D. Heermann, *Monte Carlo Simulation in Statistical Physics* (Springer-Verlag, Berlin, 1988); K. Eichhorn and K. Binder, *Z. Phys. B* **99**, 413 (1996).
 - [13] The same method has been used in the study of the relaxation behavior of the driven Ising model on regular lattices in finite dimensions. See Z. Nédá, *Phys. Lett. A* **210**, 125 (1996); M. Acharyya, *Phys. Rev. E*, **56**, 1234 (1997); **56**, 2407 (1997); S.W. Sides, P.A. Rikvold, and M.A. Novotny, *ibid.* **59**, 2710 (1999).
 - [14] B.J. Kim, M.-S. Choi, P. Minnhagen, G.S. Jeon, H.J. Kim, and M.Y. Choi, *Phys. Rev. B* **63**, 104506 (2001).

- [15] H. Hong, M.Y. Choi, and B.J. Kim, Phys. Rev. E **65**, 026139 (2002).
- [16] Z. Gao, B. Hu, and G. Hu, Phys. Rev. E **65**, 016209 (2001); M. Kuperman and D. Zanette, e-print cond-mat/0111289.